Some results on $N(k)$-contact metric manifolds

Mustafa Altın[1], Halil İbrahim Yoldaş[2] and İnan Ünal[3]

[1]Technical Sciences Vocational School, Bingöl University, Turkey
[2]Department of Management Information Systems, Adana Alparslan Turkes Science and Technology University, Turkey
[3]Department of computer Engineering, Munzur University, Turkey
E-mail: maltin@bingol.edu.tr[1], hibrahimyoldas@gmail.com[2], inanunal@munzur.edu.tr[3]

Abstract

In this study, some geometric properties of $N(k)$-contact metric manifolds, which are important class of contact manifolds, have been investigated by using a special connection (CY-connection). First, we give some fundamental results on $N(k)$-contact metric manifolds admitting CY-connection. Then, we obtain curvature properties of such manifolds. We prove that an $N(k)$-contact metric manifold admitting $R^\star(\xi, X).R^\star = 0$ condition is an $N(-\frac{1}{4})-$contact metric manifold, where $R^\star$ is the Riemannian curvature tensor of CY-connection. Also, we examine an $N(k)$-contact metric manifold admitting CY-connection under $W^\star(\xi, X).S^\star = 0$ condition for generalized quasi-conformal curvature tensor $W^\star$ of CY-connection. Finally, we consider a 3-dimensional $N(k)$-contact metric manifold admitting CY-connection.

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1 Introduction

$N(k)$-contact metric manifolds are almost contact metric manifolds which have a tangent bundle with a nullity condition. These kind of manifolds have attracted the attention of many geometers because of their important geometric properties. An $N(k)$-contact metric manifold is reduced to a Sasakian manifold if $k = 1$. Thus, $N(k)$-contact metric manifolds can be viewed as a kind of general contact manifolds, which include some classes of contact manifolds. An important example of $N(k)$-contact metric manifolds was given in [6]. In the same article, it has been proven that an $N(k)$-contact metric manifold is locally isometric to this example under the condition $Z.Z = 0$ (here, Z is concircular curvature tensor). In [11], the authors calculated the curvatures of this example and they showed that, under various special conditions, an $N(k)$-contact metric manifold is locally isometric to this example. The Riemann geometry of $N(k)$-contact metric manifolds has been studied by many researchers, some of which are: [15, 20, 10, 14, 3, 4, 9, 12, 1, 21].

The differential geometry of a Riemannian manifold is studied by considering its tangent vectors. Here, the basic tool is the Levi-Civita connection, which allows us to calculus on the manifold. The Levi-Civita connection is a torsion-free affine connection that makes the metric parallel. While examining the Riemann geometry of the manifolds, transformations with affine connection properties can be defined. These transformations correspond to connections that fail to preserve the metric or have torsion. These resulting connections are used extensively in the examination of Riemannian geometry of manifolds and provide important geometric results. Such a connection is expressed by following equation;

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$$\mathcal{D}_X Y = \nabla_X Y + H(X,Y)$$

for all vector fields $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection and H is a $(1,2)$-type tensor field. Based on different properties of $\mathcal{D}$, this connection is named as semi-symmetric metric, semi-symmetric non-metric, semi-symmetric quarter-metric etc. In [18], a generalization of such connections was given. Here, the notation $\mathcal{D}$ is used to state the generality of connections and $\widetilde{\nabla}, \widehat{\nabla}$ etc. will use for state special connections. Such connections on Riemannian manifolds with different structure have been studied by several researchers, such as in [13, 2, 19, 16]. In [8] Chaubey and Yıldız have given a definition of a new type of semi-symmetric non-metric connection. They have proved that such connection on a Riemannian manifold is projectively invariant under certain conditions. Also, they have obtained some results on Riemannian manifolds admitting this new connection. We will recall this connection by CY-connection.

In this paper, we study on $N(k)$-contact metric manifolds admitting CY-connection. Our aim is to examine the geometric properties of $N(k)$-contact metric manifolds admitting CY-connection. Classifications of these manifolds is an important notion in the contact Riemannian geometry. By this way, in this study we obtain several classifications. The paper is organized as follow; In Section 1 we give basic facts and we obtain curvature properties in the next section. In Section 4, we investigate $N(k)$-contact metric manifolds under the special condition $R^\star(\xi, X).R^\star = 0$, where $R^\star$ is the Riemannian curvature of CY-connection. Finally, we examine generalized quasi-conformal curvature tensor on $N(k)$-contact metric manifolds admitting CY-connection and we present an example.

2 Preliminaries

In this section, we shall give some essential notions and formulas which will be used later. For more details, we refer to [5].

An almost contact metric manifold is a $(2n+1)$-dimensional differentiable manifold M along with an almost contact metric structure (φ, ξ, η, g) such that ξ is a vector field of type $(0,1)$, 1-form η is the g-dual of ξ of type $(1,0)$ and φ is a tensor field of type $(1,1)$ on M and the Riemannian metric g satisfies the following relations:

$$\eta(\xi) = 1, \quad \varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \tag{2.4}$$

$$\eta(X) = g(X, \xi) \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$. Furthermore, an almost contact metric manifold M is called a contact metric manifold if it satisfies

$$\Phi(X,Y) = d\eta(X,Y).$$

for any $X, Y \in \Gamma(TM)$. Here, Φ is the fundamental $2-$form of M which is given by

$$\Phi(X,Y) = g(X, \varphi Y).$$

An almost contact metric manifold M together with the tensor field N_φ is called a normal contact metric manifold such that

$$N_\varphi + 2d\eta \otimes \xi = 0,$$

where N_φ is the Nijenhuis tensor field of φ and is defined by

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] + \varphi^2[X,Y] - \varphi[X,\varphi Y] - \varphi[\varphi X, Y]$$

for all $X, Y \in \Gamma(TM)$. A normal contact metric manifold M is called Sasakian. An almost contact metric manifold M is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

For a Sasakian manifold, we also have

$$\begin{aligned} \nabla_X \xi &= -\varphi X, \\ R(X,Y)\xi &= \eta(Y)X - \eta(X)Y. \end{aligned}$$

In [7], Blair et al. introduced the (k,μ)-nullity distribution on contact metric manifolds which satisfies

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) &= \{Z \in T_pM | R(X,Y)Z \\ &= (kI + \mu h)(g(Y,Z)X - g(X,Z)Y)\}, \end{aligned} \tag{2.6}$$

where $(k,\mu) \in \mathbb{R}^2$, I is an identity map and h is the tensor field of type $(1,1)$ defined by $h = \dfrac{1}{2}\pounds_\xi \varphi$. For such a tensor field, the followings are satisfied:

$$h\xi = 0, \tag{2.7}$$

$$\eta(hX) = 0, \tag{2.8}$$

$$h\varphi + \varphi h = 0, \tag{2.9}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{2.10}$$

$$g(hX,Y) = g(X,hY). \tag{2.11}$$

A contact metric manifold M is called a (k,μ)-contact metric manifold, if ξ belongs to (k,μ)-nullity distribution $N(k,\mu)$. If μ vanishes identically in (2.6), then the (k,μ)-nullity distribution $N(k,\mu)$ reduces to k-nullity distribution $N(k)$ and is given by [17]

$$N(k) : p \to N_p(k) = \{Z \in T_pM | R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}.$$

Also, if $\xi \in N(k)$, then a contact metric manifold M is called an $N(k)$-contact metric manifold [17]. If $k = 1$,then $N(k)$-contact metric manifold is Sasakian. If $k = 0$, then the manifold is locally isometric to the product $E^{n+1} \times S^4$ for $n > 1$ and flat for $n = 1$ [6]. On an $N(k)$-contact metric manifold, the followings are satisfied [6]:

$$
\begin{aligned}
h^2 &= (k-1)\varphi^2, && (2.12)\\
(\nabla_X \varphi)Y &= g(X+hX,Y)\xi - \eta(Y)(X+hX), && (2.13)\\
R(X,Y)\xi &= k(\eta(Y)X - \eta(X)Y), && (2.14)\\
R(\xi,X)Y &= k(g(X,Y)\xi - \eta(Y)X), && (2.15)\\
S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) && (2.16)\\
&\quad +[2nk-2(n-1)]\eta(X)\eta(Y), \quad n \geq 1 &&\\
OX &= 2(n-1)X + 2(n-1)hX + [2nk-2(n-1)]\eta(X)\xi, && (2.17)\\
S(X,\xi) &= 2nk\eta(X), && (2.18)\\
Q\xi &= 2nk\xi, && (2.19)\\
r &= 2n(2n-2+k), && (2.20)
\end{aligned}
$$

where r stands for the scalar curvature, S is the Ricci tensor and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

In 2016, Baishya and Chowdhury [3] introduced the generalized quasi-conformal curvature tensor by

$$
\begin{aligned}
W(X,Y)Z &= R(X,Y)Z + a[S(Y,Z)X - S(X,Z)Y]\\
&\quad +b[g(Y,Z)QX - g(X,Z)QY] \qquad (2.21)\\
&\quad -\frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)[g(Y,Z)X - g(X,Z)Y]
\end{aligned}
$$

for any $X,Y,Z \in \Gamma(TM)$, where a,b and c are real constants.

On the other hand, a Riemannian manifold (M,g) is called η-Einstein if there exists two real constants a and b such that the Ricci tensor field S of M satisfies

$$S = ag + b\eta \otimes \eta.$$

If the constant b is equal to zero, then M is called an Einstein manifold.

3 Curvature properties of $N(k)$-contact metric manifolds with respect to CY-connection

In this section, we deal with $N(k)$-contact metric manifolds with respect to CY-connection and obtained some important results related to Riemannian curvature tensor and Ricci tensor of these manifolds.

In [8], the authors defined a type of non-symmetric an non-metric connection. We recall this connection by CY-connection. The definition of CY-connection on $N(k)$-contact metric manifolds is given by the following.

Let (M,φ,ξ,η,g) be an almost contact metric manifold and ∇ be a Levi-Civita connection on M. For any $X,Y \in \Gamma(TM)$, let define a map;

$$
\begin{gathered}
\nabla^{\star} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)\\
\nabla^{\star}_X Y = \nabla_X Y + \frac{1}{2}(\eta(Y)X - \eta(X)Y). \qquad (3.1)
\end{gathered}
$$

Covariant derivation of metric with respect to CY-connection, and the torsion of CY-connection are given by

$$(\nabla^\star_X g)(Y,Z) = \eta(X)g(Y,Z) - \frac{1}{2}\{\eta(Y)g(X,Z) - \eta(Z)g(X,Y)\}$$

and

$$T^\star(X,Y) = \eta(Y)X - \eta(X)Y$$

for any $X, Y, Z \in \Gamma(TM)$, where $T^\star$ stands for the torsion tensor of connection $\nabla^\star$. In this case we can state that $\nabla^\star$ is a non-metric and non-symmetric connection.

Proposition 3.1. Let M be an $N(k)$-contact metric manifold with respect to CY-connection $\nabla^\star$. Then, we have

$$\nabla^\star_X \xi = -\varphi X - \varphi hX + \frac{1}{2}\{X - \eta(X)\xi\}, \tag{3.2}$$

$$(\nabla^\star_X \eta)Y = -g(Y, \varphi X + \varphi hX), \tag{3.3}$$

$$(\nabla^\star_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(hX + X) - \frac{1}{2}\eta(Y)\varphi X. \tag{3.4}$$

Proof. From (2.12) and (3.1), it is clear that the equalities (3.2) and (3.3) are satisfied. Also using (2.2), (2.13) and (3.1) we get

$$\begin{aligned}
(\nabla^\star_X \varphi)Y &= \nabla^\star_X \varphi Y - \varphi(\nabla^\star_X Y)\\
&= \nabla_X \varphi Y + \frac{1}{2}(-\eta(X)\varphi Y) - \varphi(\nabla_X Y + \frac{1}{2}(\eta(Y)X - \eta(X)Y))\\
&= (\nabla_X \varphi)Y + \frac{1}{2}(-\eta(X)\varphi Y) - \varphi(\nabla_X Y + \frac{1}{2}(\eta(Y)X - \eta(X)Y))\\
&= g(X + hX, Y)\xi - \eta(Y)(hX + X) - \frac{1}{2}\eta(Y)\varphi X
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Thus, the proof is completed. Q.E.D.

Let M be an $N(k)$-contact metric manifold with respect to CY-connection $\nabla^\star$. Then, the Riemannian curvature tensor $R^\star$ of M with respect to CY-connection $\nabla^\star$ is given by

$$R^\star(X,Y)Z = \nabla^\star_X \nabla^\star_Y Z - \nabla^\star_Y \nabla^\star_X Z - \nabla^\star_{[X,Y]} Z \tag{3.5}$$

for any $X, Y, Z \in \Gamma(TM)$. Using the equations (2.5), (2.10) and (3.1), we derive that

$$\begin{aligned}
\nabla^\star_X \nabla^\star_Y Z &= \nabla_X \nabla_Y Z + \frac{1}{2}\{\eta(\nabla_Y Z)X + \eta(\nabla_X Z)Y - \eta(\nabla_X Y)Z\\
&\quad -\eta(X)\nabla_Y Z + \eta(Z)\nabla_X Y - \eta(Y)\nabla_X Z\}\\
&\quad +\frac{1}{2}\{g(Y, \varphi X + \varphi hX)Z - g(Z, \varphi X + \varphi hX)Y\}\\
&\quad +\frac{1}{4}\{\eta(X)\eta(Y)Z - \eta(X)\eta(Z)Y\}.
\end{aligned} \tag{3.6}$$

Interchanging the roles of X and Y in (3.6) gives

$$\begin{aligned}\nabla^\star_Y \nabla^\star_X Z &= \nabla_Y \nabla_X Z + \frac{1}{2}\{\eta(\nabla_X Z)Y + \eta(\nabla_Y Z)X - \eta(\nabla_Y X)Z \\ &\quad -\eta(Y)\nabla_X Z + \eta(Z)\nabla_Y X - \eta(X)\nabla_Y Z\} \\ &\quad +\frac{1}{2}\{g(X, \varphi Y + \varphi hY)Z - g(Z, \varphi Y + \varphi hY)X\} \\ &\quad +\frac{1}{4}\{\eta(Y)\eta(X)Z - \eta(Y)\eta(Z)X\}.\end{aligned} \tag{3.7}$$

Also, making use of (3.1) one has

$$\begin{aligned}\nabla^\star_{[X,Y]} Z = \nabla_{[X,Y]} Z + \frac{1}{2}\{\eta(Z)\nabla_X Y - \eta(Z)\nabla_Y X \\ +\eta(\nabla_Y X)Z - \eta(\nabla_X Y)Z\}.\end{aligned} \tag{3.8}$$

With the help of (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned}R^\star(X,Y)Z &= R(X,Y)Z + \frac{1}{4}\eta(Z)\{\eta(Y)X - \eta(X)Y\} + g(Y, \varphi X)Z \\ &\quad +\frac{1}{2}\{g(Z, \varphi Y + \varphi hY)X - g(Z, \varphi X + \varphi hX)Y\},\end{aligned} \tag{3.9}$$

where R denotes the Riemannian curvature tensor of M with respect to Levi-Civita connection. Taking inner product of (3.9) with arbitary vector field W yields

$$\begin{aligned}R^\star(X,Y,Z,W) &= R(X,Y,Z,W) + \frac{1}{4}\eta(Z)\{\eta(Y)g(X,W) - \eta(X)g(Y,Z)\} \\ &\quad +\frac{1}{2}\{g(Z, \varphi Y + \varphi hY)g(X,W) - g(Z, \varphi X + \varphi hX)g(Z,W)\} \\ &\quad +g(Y, \varphi X)g(Z,W),\end{aligned} \tag{3.10}$$

where $R^\star(X,Y,Z,W) = g(R^\star(X,Y)Z,W)$ and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$. In this case, we can write

$$R^\star(X,Y,Z,W) = -R^\star(Y,X,Z,W)$$

and

$$\begin{aligned}R^\star(X,Y,Z,W) + R^\star(Y,Z,X,W) + R^\star(Z,X,Y,W) = \\ 2\{g(X,\varphi Z)g(Y,W) + g(Z,\varphi Y)g(X,W) + g(Y,\varphi X)g(Z,W)\}.\end{aligned}$$

Let $\{e_1, e_2, ..., e_{2n+1} = \xi\}$ be an orthonormal basis of the tangent space T_pM, at each point $p \in M$. Then, the Ricci tensor $S^\star$ of M with respect to CY-connection $\nabla^\star$ is defined by

$$S^\star(Y,Z) = \sum_{i=1}^{2n+1} R^\star(e_i, Y, Z, e_i) \tag{3.11}$$

for any $Y, Z \in \Gamma(TM)$. Putting $X = W = e_i$ in (3.10) and summing over i $(i = 1, 2, ..., 2n + 1)$ we get

$$S^{\star}(Y, Z) = S(Y, Z) + \frac{1}{2}g(\varphi Y - \varphi hY, Z) + \frac{1}{4}(g(Y, Z) - \eta(Y)\eta(Z)) \tag{3.12}$$

and hence

$$Q^{\star}Y = QY + \frac{1}{2}(\varphi Y - \varphi hY) + \frac{1}{4}(Y - \eta(Y)\xi). \tag{3.13}$$

Here, $Q^{\star}$ and Q are the Ricci operator of M with respect to $\nabla^{\star}$ and ∇, respectively. It follows from (3.12) that we have

$$r^{\star} = r + \frac{n}{2} \tag{3.14}$$

where $r^{\star}$ and r represent the scalar curvature of M with respect to $\nabla^{\star}$ and ∇, respectively.

Now, we are in a position to state the following.

Proposition 3.2. Let M be an $N(k)$-contact metric manifold with respect to CY-connection $\nabla^{\star}$. Then,

i) The Riemann curvature tensor $R^{\star}$ of M with respect to $\nabla^{\star}$ is given by (3.9).

ii) The Ricci tensor $S^{\star}$ of M with respect to $\nabla^{\star}$ is given by (3.12) and is not a symmetric tensor.

iii) The Ricci operator $Q^{\star}$ of M with respect to $\nabla^{\star}$ is given by (3.13).

iv) The scalar curvature $r^{\star}$ of M with respect to $\nabla^{\star}$ is given by (3.14).

Proposition 3.3. Let M be an $N(k)$-contact metric manifold with respect to CY-connection $\nabla^{\star}$. Then, we have the followings:

$$\begin{aligned}
R^{\star}(X, Y)\xi &= (k + \frac{1}{4})\{\eta(Y)X - \eta(X)Y\} + g(Y, \varphi X)\xi, && (3.15)\\
R^{\star}(\xi, X)\xi &= (k + \frac{1}{4})\{\eta(Y)\xi - Y\}, && (3.16)\\
R^{\star}(\xi, X)Y &= k\{g(X, Y)\xi - \eta(Y)X\} + \frac{1}{4}\eta(Y)\{\eta(X)\xi - X\} && (3.17)\\
&\quad + \frac{1}{2}g(Y, \varphi X + \varphi hX)\xi, \\
\eta(R^{\star}(\xi, X)Y) &= k\{g(X, Y) - \eta(Y)\eta(X)\} + \frac{1}{2}g(Y, \varphi X + \varphi hX), && (3.18)\\
S^{\star}(X, \xi) &= 2nk\eta(X), && (3.19)\\
S^{\star}(\xi, \xi) &= 2nk, && (3.20)\\
Q^{\star}\xi &= 2nk\xi && (3.21)
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

4 $N(k)$-contact metric manifolds satisfying $R^\star(\xi,X).R^\star=0$

Let us assume that an $N(k)$-contact metric manifold M with respect to CY-connection $\nabla^\star$ satisfies $(R^\star(\xi,X).R^\star)(Y,Z)U=0$, namely

$$\begin{aligned}&R^\star(\xi,X).R^\star(Y,Z)U-R^\star(R^\star(\xi,X)Y,Z)U\\&-R^\star(Y,R^\star(\xi,X)Z)U-R^\star(Y,Z)R^\star(\xi,X)U=0\end{aligned}\tag{4.1}$$

for any $X,Y,Z,U\in\Gamma(TM)$. Substituting Y for ξ in (4.1), then we write

$$\begin{aligned}&R^\star(\xi,X).R^\star(\xi,Z)U-R^\star(R^\star(\xi,X)\xi,Z)U\\&-R^\star(\xi,R^\star(\xi,X)Z)U-R^\star(\xi,Z)R^\star(\xi,X)U=0.\end{aligned}\tag{4.2}$$

For the first term of the equality (4.2), using (3.17) we have

$$\begin{aligned}R^\star(\xi,X)R^\star(\xi,Z)U&=k\{g(X,R^\star(\xi,Z)U)\xi-\eta(R^\star(\xi,Z)U)X\}\\&+\frac{1}{4}\eta(R^\star(\xi,Z)U)\{\eta(X)\xi-X\}+\frac{1}{2}g(R^\star(\xi,Z)U,\varphi X+\varphi hX)\xi.\end{aligned}\tag{4.3}$$

By virtue of the equalities (2.2), (2.5), (2.7), (3.17), (3.18) and (4.3), we get

$$\begin{aligned}R^\star(\xi,X)R^\star(\xi,Z)U&=(k^2+\frac{k}{4})\Big\{g(U,Z)\eta(X)\xi-g(X,Z)\eta(U)\xi\\&-g(U,Z)X+\eta(U)\eta(Z)X\Big\}+(\frac{k}{2}+\frac{1}{8})\Big\{g(U,\varphi Z+\varphi hZ)\eta(X)\xi\\&-g(U,\varphi Z+\varphi hZ)X-g(Z,\varphi X+\varphi hX)\eta(U)\xi\Big\}.\end{aligned}\tag{4.4}$$

For the second term of the equality (4.2), using (3.16) one has

$$R^\star(R^\star(\xi,X)\xi,Z)U=(k+\frac{1}{4})\eta(X)R^\star(\xi,Z)U-(k+\frac{1}{4})R^\star(X,Z)U.\tag{4.5}$$

From (3.9), (3.17) and (4.5), we obtain

$$\begin{aligned}&R^\star(R^\star(\xi,X)\xi,Z)U=(k^2+\frac{k}{4})\Big\{g(U,Z)\eta(X)\xi-\eta(X)\eta(U)Z\Big\}\\&(\frac{k}{4}+\frac{1}{16})\Big\{\eta(X)\eta(U)\eta(Z)\xi-\eta(U)\eta(Z)X\Big\}-(k+\frac{1}{4})\Big\{R(X,Z)U\\&g(Z,\varphi X)U\Big\}+(\frac{k}{2}+\frac{1}{8})\Big\{g(U,\varphi Z+\varphi hZ)\eta(X)\xi\\&-g(U,\varphi Z+\varphi hZ)X+g(U,\varphi X+\varphi hX)Z\Big\}.\end{aligned}\tag{4.6}$$

For the third term of the equality (4.2), making use of (3.17) we have

$$\begin{aligned}R^\star(\xi,R^\star(\xi,X)Z)U\quad=\quad&k\{g(U,R^\star(\xi,X)Z)\xi-\eta(U)R^\star(\xi,X)Z\}\\&+\frac{1}{4}\eta(U)\{\eta(R^\star(\xi,X)Z)\xi-R^\star(\xi,X)Z\}\\&+\frac{1}{2}g(U,\varphi(R^\star(\xi,X)Z)+\varphi h(R^\star(\xi,X)Z))\xi.\end{aligned}\tag{4.7}$$

If we employ the equalities (2.2), (2.7), (3.17) and (3.18) in (4.7), we infer that

$$\begin{aligned} R^{\star}(\xi, R^{\star}(\xi, X)Z)U &= (k^2 + \frac{k}{4})\Big\{ - g(U, X)\eta(Z)\xi + \eta(U)\eta(Z)X\Big\} \\ &\quad -(\frac{k}{4} + \frac{1}{16})\Big\{\eta(X)\eta(U)\eta(Z)\xi - \eta(U)\eta(Z)X\Big\} \\ &\quad -(\frac{k}{2} + \frac{1}{8})g(U, \varphi X + \varphi hX)\eta(Z)\xi. \end{aligned} \tag{4.8}$$

For the fourth term of the equality (4.2), interchanging the roles of X and Z in (4.4) we find

$$\begin{aligned} &R^{\star}(\xi, Z)R^{\star}(\xi, X)U = (k^2 + \frac{k}{4})\Big\{g(U, X)\eta(Z)\xi - g(Z, X)\eta(U)\xi \\ &-g(U, X)Z + \eta(U)\eta(X)Z\Big\} + (\frac{k}{2} + \frac{1}{8})\Big\{g(U, \varphi X + \varphi hX)\eta(Z)\xi \\ &\qquad -g(U, \varphi X + \varphi hX)Z - g(X, \varphi Z + \varphi hZ)\eta(U)\xi\Big\}. \end{aligned} \tag{4.9}$$

Setting (4.4), (4.6), (4.8) and (4.9) in (4.2), we deduce

$$\begin{aligned} &(k^2 + \frac{k}{4})\Big\{g(U, X)Z - g(U, Z)X\Big\} + (k + \frac{1}{4})\Big\{R(X, Z)U + g(Z, \varphi X)U\Big\} \\ &+(\frac{k}{2} + \frac{1}{8})\Big\{g(X, \varphi Z + \varphi hZ)\eta(U)\xi - g(Z, \varphi X + \varphi hX)\eta(U)\xi\Big\} = 0. \end{aligned} \tag{4.10}$$

Moreover, using (2.4) and (2.9) in (4.10) yields

$$(k + \frac{1}{4})\Big\{kg(U, X)Z - kg(U, Z)X + R(X, Z)U - g(X, \varphi Z)U \tag{4.11}$$
$$+g(X, \varphi Z)\eta(U)\xi\Big\} = 0.$$

Taking inner product of (4.11) with arbitrary vector field T gives

$$(k + \frac{1}{4})\Big\{kg(U, X)g(Z, T) - kg(U, Z)g(X, T) + R(X, Z, U, T) \tag{4.12}$$
$$-g(X, \varphi Z)g(U, T) + g(X, \varphi Z)\eta(U)\eta(T)\Big\} = 0.$$

Contracting over X and T in (4.12), we obtain

$$(k + \frac{1}{4})(S(U, Z) - 2nkg(U, Z) - g(U, \varphi Z)) = 0. \tag{4.13}$$

Now, if $k \neq -\frac{1}{4}$, then we have

$$S(U, Z) = 2nkg(U, Z) + g(U, \varphi Z). \tag{4.14}$$

Since the Ricci tensor S with respect to Levi-Civita connection ∇ is symmetric in U and Z, we write

$$S(Z, U) = 2nkg(Z, U) + g(Z, \varphi U). \tag{4.15}$$

Subtracting (4.15) from (4.14) and using (2.4) yields $g(Z,\varphi U)=0$, which is not possible in contact metric manifolds. In this case, from (4.13) we have $k=-\frac{1}{4}$.

Now, we can give the following theorem.

Theorem 4.1. Let M be an $N(k)$-contact metric manifold M with respect to CY-connection $\nabla^\star$ satisfies $R^\star(\xi,X).R^\star=0$. Then, M is an $N(-\frac{1}{4})-$contact metric manifold.

5 $N(k)$-contact metric manifolds satisfying $W^\star(\xi,X).S^\star=0$

Let M be an $N(k)$-contact metric manifold with respect to CY-connection $\nabla^\star$. In an $N(k)$-contact metric manifold M, the definition of the generalized quasi-conformal curvature tensor $W^\star$ is given by

$$\begin{aligned} W^\star(X,Y)Z &= R^\star(X,Y)Z+a[S^\star(Y,Z)X-S^\star(X,Z)Y] \\ &\quad +b[g(Y,Z)Q^\star X-g(X,Z)Q^\star Y] \\ &\quad -\frac{cr^\star}{2n+1}(\frac{1}{2n}+a+b)[g(Y,Z)X-g(X,Z)Y] \end{aligned} \tag{5.1}$$

for any $X,Y,Z\in\Gamma(TM)$. Using (2.20), (3.9), (3.12), (3.13) and (3.14) in (5.1) we have

$$\begin{aligned} W^\star(X,Y)Z &= R(X,Y)Z+\frac{1}{4}\eta(Z)\{\eta(Y)X-\eta(X)Y\}+g(Y,\varphi X)Z \\ &\quad +\frac{1}{2}\{g(Z,\varphi Y+\varphi hY)X-g(Z,\varphi X+\varphi hX)Y\}+a\Big\{S(Y,Z)X \\ &\quad +\frac{1}{2}g(\varphi Y-\varphi hY,Z)X+\frac{1}{4}g(Y,Z)X-\frac{1}{4}\eta(Y)\eta(Z))X-S(X,Z)Y \\ &\quad -\frac{1}{2}g(\varphi X-\varphi hX,Z)Y-\frac{1}{4}g(X,Z)Y+\frac{1}{4}\eta(X)\eta(Z))Y\Big\} \\ &\quad +b\Big\{g(Y,Z)\{QX+\frac{1}{2}(\varphi X-\varphi hX)+\frac{1}{4}(X-\eta(X)\xi)\} \\ &\quad -g(X,Z)\{QY+\frac{1}{2}(\varphi Y-\varphi hY)+\frac{1}{4}(Y-\eta(Y)\xi)\}\Big\} \\ &\quad -\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\{g(Y,Z)X-g(X,Z)Y\Big\}. \end{aligned} \tag{5.2}$$

Putting $X=\xi$ in (5.2) and using (2.2), (2.7), (2.15), (2.18), (2.19) and (2.20) gives

$$\begin{aligned} W^\star(\xi,Y)Z &= k(g(Y,Z)\xi-\eta(Z)Y)+\frac{1}{4}\eta(Z)\{\eta(Y)\xi-Y\} \\ &\quad +\frac{1}{2}\{g(Z,\varphi Y+\varphi hY)\xi\}+a\Big\{S(Y,Z)\xi+\frac{1}{2}g(\varphi Y-\varphi hY,Z)\xi \\ &\quad +\frac{1}{4}g(Y,Z)\xi-\frac{1}{4}\eta(Y)\eta(Z)\xi-2nk\eta(Z)Y\Big\}+b\Big\{2nkg(Y,Z)\xi \\ &\quad -\eta(Z)QY-\eta(Z)\{\frac{1}{2}(\varphi Y-\varphi hY)+\frac{1}{4}(Y-\eta(Y)\xi)\}\Big\} \\ &\quad -\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\{g(Y,Z)\xi-\eta(Z)Y\Big\}. \end{aligned} \tag{5.3}$$

Replacing Z by ξ in (5.3) and from (2.1), (2.2), (2.17) and (2.18), we get

$$\begin{aligned} W^\star(\xi,Y)\xi &= \Big\{k+\frac{1}{4}+2nka+2(n-1)b+\frac{b}{4} \\ &-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}\eta(Y)\xi-\Big\{k+\frac{1}{4}+2nka \\ &+2(n-1)b+\frac{b}{4}-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}Y \\ &-2(n-1)bhY-\frac{b}{2}\varphi Y+\frac{b}{2}\varphi hY. \end{aligned} \tag{5.4}$$

Also, applying η to the both sides of (5.3) and using (2.2), (2.8), (2.18) and (2.19) one has

$$\begin{aligned} \eta(W^\star(\xi,Y)Z) &= \Big\{k+\frac{a}{4}+2nkb-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\} \\ &g(Y,Z)-\Big\{k+\frac{a}{4}+2nk(a+b)-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\} \\ &\eta(Y)\eta(Z)+\frac{1}{2}g(Z,\varphi Y+\varphi hY)+\frac{a}{2}g(\varphi Y-\varphi hY,Z)+aS(Y,Z). \end{aligned} \tag{5.5}$$

Now, suppose that an $N(k)$-contact metric manifold M with respect to CY-connection $\nabla^\star$ satisfies $(W^\star(\xi,Y).S^\star)(Z,T)=0$, that is,

$$S^\star(W^\star(\xi,Y)Z,T)+S^\star(Z,W^\star(\xi,Y)T)=0 \tag{5.6}$$

for any $Y,Z,T\in\Gamma(TM)$. Taking ξ instead of T in (5.6) and using (3.19), we write

$$2nk\eta(W^\star(\xi,Y)Z)+S^\star(Z,W^\star(\xi,Y)\xi)=0. \tag{5.7}$$

For the second term of the equality (5.7), using (3.16) we arrive at

$$\begin{aligned} S^\star(Z,W^\star(\xi,Y)\xi) &= S(Z,W^\star(\xi,Y)\xi)+\frac{1}{2}g(\varphi Z-\varphi hZ,W^\star(\xi,Y)\xi) \\ &+\frac{1}{4}g(Z,W^\star(\xi,Y)\xi)-\frac{1}{4}\eta(Z)\eta(W^\star(\xi,Y)\xi). \end{aligned} \tag{5.8}$$

In view of the equalities (2.2), (2.7), (2.18), (5.4) and (5.8) and after long calculations, we obtain

$$\begin{aligned} S^\star(Z,W^\star(\xi,Y)\xi) &= 2nk\Big\{k+\frac{1}{4}+2nka+2(n-1)b+\frac{b}{4} \\ &-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}\eta(Y)\eta(Z)-\Big\{k+\frac{1}{4}+2nka \\ &+2(n-1)b+\frac{b}{4}-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}S(Y,Z) \\ &-2(n-1)bS(Z,hY)-\frac{b}{2}S(Z,\varphi Y)+\frac{b}{2}S(Z,\varphi hY)-\frac{1}{2}\Big\{k+\frac{1}{4} \\ &+2nka+2(n-1)b+\frac{b}{4}-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\} \end{aligned} \tag{5.9}$$

$$
\begin{aligned}
&+g(\varphi Z-\varphi hZ,Y)-(n-1)bg(\varphi Z-\varphi hZ,hY)-\frac{b}{4}g(\varphi Z-\varphi hZ,\varphi Y)\\
&+\frac{b}{4}g(\varphi Z-\varphi hZ,\varphi hY)+\frac{1}{4}\Big\{k+\frac{1}{4}+2nka+2(n-1)b+\frac{b}{4}\\
&-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}\eta(Y)\eta(Z)-\frac{1}{4}\Big\{k+\frac{1}{4}+2nka\\
&+2(n-1)b+\frac{b}{4}-\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}g(Z,Y)\\
&-\frac{b(n-1)}{2}g(Z,hY)-\frac{b}{8}g(\varphi Y-\varphi hY,Z).
\end{aligned}
$$

Using the equalities (5.5) and (5.9) in (5.7), we derive

$$
\begin{aligned}
&\Big\{-4nkb(nk-n+1)+\frac{nk}{2}+\frac{nkb}{2}+\frac{k}{4}+\frac{1}{16}+\frac{(n-1)b}{2}+\frac{b}{16}\\
&-\frac{1}{4}\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}\eta(Y)\eta(Z)+\Big\{-k-\frac{1}{4}\\
&-2(n-1)b-\frac{b}{4}+\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}S(Y,Z)\\
&+\Big\{-\frac{k}{4}-\frac{1}{16}-\frac{(n-1)b}{2}+\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)(\frac{1}{4}-2nk)\\
&-\frac{b}{16}+2nk^2+4n^2k^2b\Big\}g(Y,Z)+nkg(Z,\varphi Y+\varphi hY)\\
&+(nka-\frac{b}{8})g(\varphi Y-\varphi hY,Z)+\Big\{-\frac{k}{2}-\frac{1}{8}-nka-(n-1)b-\frac{b}{2}\\
&+\frac{1}{2}\frac{cn(8n-7+4k)}{4n+2}(\frac{1}{2n}+a+b)\Big\}g(\varphi Z-\varphi hZ,Y)-\frac{b}{2}S(Z,\varphi Y)\\
&-(n-1)bg(\varphi Z-\varphi hZ,hY)-\frac{b}{4}g(\varphi Z-\varphi hZ,\varphi Y)+\frac{b}{2}S(Z,\varphi hY)\\
&+\frac{b}{4}g(\varphi Z-\varphi hZ,\varphi hY)-2(n-1)bS(Z,hY)=0.
\end{aligned}
\tag{5.10}
$$

Now, we are ready to give the following result.

Theorem 5.1. Let M be an $N(k)$-contact metric manifold M with respect to CY-connection $\nabla^\star$ satisfies $W^\star(\xi,X).S^\star=0$. Then, the Ricci tensor S of M with respect to ∇ is of form the equation (5.10).

Example 5.2. [11] Let consider a three-dimensional manifold

$$M=\{(x,y,z)\in\mathbb{R}^3,(x,y,z)\neq(0,0,0)\},$$

where (x,y,z) are the Cartesian coordinates in $\mathbb{R}^3$. Let e_1, e_2 and e_3 be linearly independent vector fields in $\mathbb{R}^3$ which satisfies

$$[e_1,e_2]=(1+\lambda)e_3,\quad [e_1,e_3]=-(1-\lambda)e_2\quad \textit{and}\quad [e_2,e_3]=2e_1,$$

where λ is a real number and g be a Riemannian metric defined by

$$g(e_i, e_i) = 1, \qquad g(e_i, e_j) = 0 \qquad for \qquad i \neq j.$$

Let define a 1-form η and $(1,1)-$tensor field φ as follow

$$g(Z, e_1) = \eta(Z), \quad \varphi(e_2) = e_3, \quad \varphi(e_3) = -e_2, \quad \varphi(e_1) = 0$$

for any $Z \in \Gamma(TM)$. Then we define a map h by

$$he_1 = 0, \qquad he_2 = \lambda e_2, \qquad and \qquad he_3 = -\lambda e_3.$$

By using Koszul's formula for the Riemannian metric g, we get

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_1} e_3 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \tag{5.11}$$

$$\nabla_{e_3} e_2 = -(1-\lambda)e_1, \qquad \nabla_{e_3} e_1 = (1-\lambda)e_2, \tag{5.12}$$

$$\nabla_{e_2} e_1 = -(1+\lambda)e_3, \qquad \nabla_{e_2} e_3 = (1+\lambda)e_1. \tag{5.13}$$

Using the above equations, we infer

$$R(e_1, e_2)e_3 = 0, \qquad R(e_1, e_3)e_2 = 0, \qquad R(e_2, e_3)e_1 = 0, \tag{5.14}$$

$$R(e_1, e_2)e_2 = (1-\lambda^2)e_1, \quad R(e_1, e_2)e_1 = -(1-\lambda^2)e_2, \tag{5.15}$$

$$R(e_1, e_3)e_3 = (1-\lambda^2)e_1, \quad R(e_1, e_3)e_1 = -(1-\lambda^2)e_3, \tag{5.16}$$

$$R(e_2, e_3)e_3 = -(1-\lambda^2)e_2, \quad R(e_2, e_3)e_2 = (1-\lambda^2)e_3. \tag{5.17}$$

Hence, M is a 3-dimensional $N(k)$-contact metric manifold with respect to Levi-Civita connection ∇. Utilizing from (5.11)-(5.13), we get

$$\nabla^\star_{e_1} e_1 = \nabla^\star_{e_2} e_2 = \nabla^\star_{e_3} e_3 = 0, \quad \nabla^\star_{e_3} e_2 = -(1-\lambda)e_1, \tag{5.18}$$

$$\nabla^\star_{e_3} e_1 = (1-\lambda)e_2 + \frac{1}{2}e_3, \quad \nabla^\star_{e_1} e_2 = -\frac{1}{2}e_2, \quad \nabla^\star_{e_1} e_3 = -\frac{1}{2}e_3, \tag{5.19}$$

$$\nabla^\star_{e_2} e_1 = -(1+\lambda)e_3 + \frac{1}{2}e_2, \quad \nabla^\star_{e_2} e_3 = (1+\lambda)e_1. \tag{5.20}$$

Due to the equalities (3.5) and (5.18)-(5.20), we find that

$$R^\star(e_1, e_2)e_3 = 0, \quad R^\star(e_1, e_3)e_2 = 0, \quad R^\star(e_2, e_3)e_1 = 0, \tag{5.21}$$

$$R^\star(e_1, e_2)e_2 = (1-\lambda^2)e_1, \quad R^\star(e_1, e_2)e_1 = -(1-\lambda^2)e_2 - \frac{1}{4}e_2, \tag{5.22}$$

$$R^\star(e_1, e_3)e_3 = (1-\lambda^2)e_1, \quad R^\star(e_1, e_3)e_1 = -(1-\lambda^2)e_3 - \frac{1}{4}e_3, \tag{5.23}$$

$$R^\star(e_2, e_3)e_3 = -(1-\lambda^2)e_2 + (\frac{1-\lambda}{2})e_3, \tag{5.24}$$

$$R^\star(e_2, e_3)e_2 = -(1-\lambda^2)e_3 - (\frac{1+\lambda}{2})e_2, \tag{5.25}$$

which satisfy (3.9). Also, from the equalities (3.11) and (5.21)-(5.25), we get

$$S^\star(e_1, e_1) = 2(1-\lambda^2), \quad S^\star(e_2, e_2) = 0, \quad S^\star(e_3, e_3) = 0, \quad S^\star(e_i, e_j) = 0$$

for all $i, j = 1, 2, 3$ $(i \neq j)$. In this case, M is a 3-dimensional $N(k)$-contact metric manifold with respect to $\nabla^\star$.

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